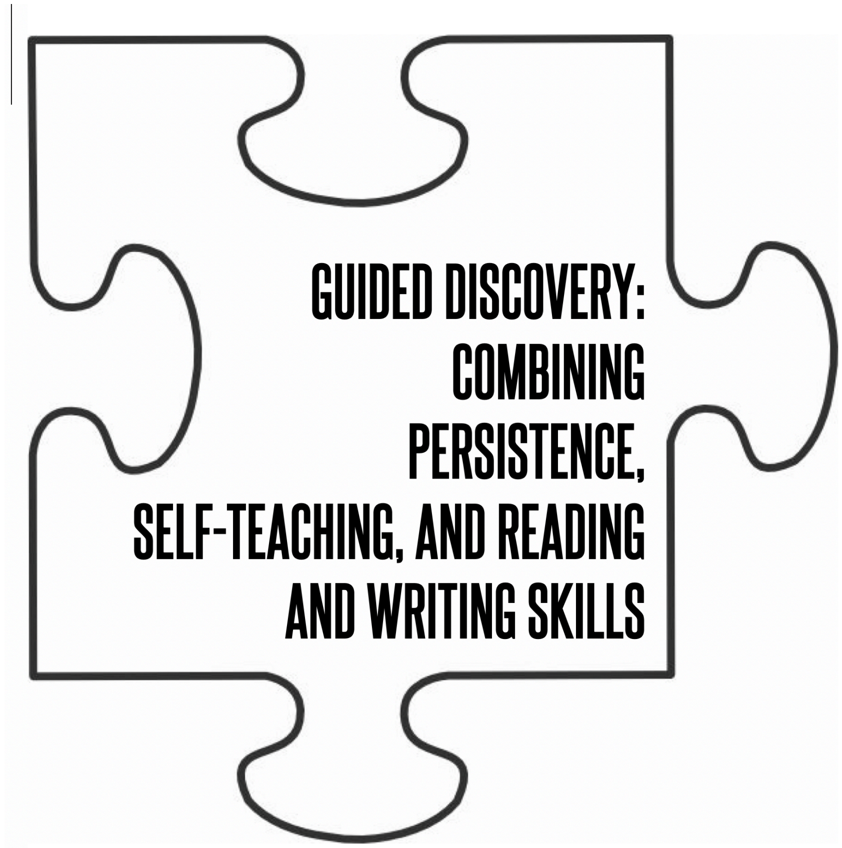

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WHAT IS A GUIDED DISCOVERY LESSON?

Guided discovery lessons have a story line that inherently engages the participant. Think about how you became engrossed in your favorite book or movie as the story line unfolded. Someone could have described the story to you in a 30-second encapsulation and you would not have been so engrossed. For example, in the movie Titanic, boy meets girl on ship, ship sinks, girl loses boy in water. Certainly, this is the formula of the Titanic story line but you cannot get emotionally and intellectually vested in this 30-second synopsis.

As the name suggests, a guided discovery lesson is guided. If you have ever filled out an IRS tax return, you have been gradually guided. An IRS form is prescriptive in nature, with no intent to have the reader “discover” anything. Guided discovery lessons also contain the all-important aspect of discovery. The thrill of discovery is an inherent part of the way real mathematics is developed and should be a part of the way school mathematics is taught. At times, due to curricular, assessment, state mandates and other influences, our mathematics courses may be devoid of this feature that positions the study of mathematics as a living, breathing science.

In a guided-discovery lesson, students sequentially uncover layers of mathematical information one step at a time and learn a new fact or procedure. The questions guide the students slowly and methodically, and it is essential that students do not skim as they read. Perhaps one of the most beneficial by–products of a guided discovery activity is that students do more than just learn the mathematics. The activity serves to convince them that they can both learn and do mathematics on their own. This is important for struggling students.

Writing a Guided Discovery Lesson

The writing strategies and activities in this resource book will be instrumental in shaping your guided discovery lesson. The essential elements of writing a guided discovery lesson include empathizing with someone who does not know the material, and then explaining it in a clear, sequential, gradual fashion.

1. Select the Content

Certain topics lend themselves better to guided discovery than others. Lengthy, convoluted developments are not conducive to the guided discovery approach. You could write a nice guided discovery lesson on deriving the area of a trapezoid, but a lesson connecting the limit of a sum to the process of integration might be too deep.

Guided-discovery activities must have a new component - something new to the reader must be discovered. A guided discovery activity is not a review sheet. It must draw upon the things students can already do, guiding them into uncharted (for them) mathematical territory.

2. Identify the Entry Conditions

What do the students need to know to be successful in your guided discovery lesson? Prior to the guided discovery lesson, make sure that the prerequisites for the activity have been met. You can assess via a quiz, a "Do Now" activity, a whole class discussion, or a review of the previous day's homework assignment. A guided discovery lesson can be sabotaged when students do not have the skills they need to proceed.

3. Handling the Objective

The objective can be explicitly stated if it doesn't undermine the surprise and therefore spoil the discovery part of the lesson. If this is the case, you can give the lesson a related introduction that is generic in nature but doesn't spoil the discovery.

4. Outline the Key Gradual Steps

Make a schematic outline of the basic structure of the lesson. It is important to first look at the topic globally before you dissect it into its component parts. Then use the writing strategies to dissect the development into gradual steps. Remember, there is no danger in being too clear.

5. Write Your Lesson

When writing your lesson, remember to be empathetic and use the writing strategies. The students do not know what you know, so write your lesson at their experience level. Successful guided discovery lessons thrive on clarity. When in doubt, err of the side of over clarification. Do not be afraid of "over developing" the step-by-step flow of the lesson. The strength in guided discovery is that the students see the concepts unfurl gradually before their eyes. The key to learning is their engagement in the material.

6. Plan your Checkpoints

Your role during the guided-discovery lesson is that of a roving coach. Reflection points are junctures at which you ask students to stop and reflect, discuss, write, and/or explain. Once a student arrives at a reflection point, he/she should call you over before progressing any further with the activity. If the remainder of the guided discovery activity is dependent upon a basic level of correctness up to that certain point, then having the student stop all work and ask for a teacher check would be critical for future success in the endeavor.

<u>7. Get a Naïve Proofreader</u>

Have a colleague or another student actually do the activity. This may help you to uncover pitfalls and unclear directions or assumptions you have made. Being your own proofreader (completing the activity in its entirety yourself) is fine as a last resort, but may not be as effective as when you have someone go through it who is further removed from the writing process. Who could serve as a naïve proofreader? Other teachers, ex-students in later grades who still come to visit you, and possibly current students. You could have a student complete the activity on his/her own at home a few days before you actually give it to the class. They will undoubtedly find junctures that you can smooth out with a revision. Guided-discovery lessons are perfect for the student looking to do independent work, and this is one way you can try them out.

<u>8. Write a Follow-Up Activity to Check for Accountability</u>

Did the students learn the material? They will work more purposefully if they know that they will be held accountable for this "independent" learning. You may wish to have the guided discovery activity be completed in cooperative groups one day, and administer an individual follow-up assessment the next day. You may decide to allow the students to use notes generated during the group activity to be used in the individual accountability assessment.

<u>9. Field Test, Critique, and Revise</u>

Keep notes about how the lesson progresses in each of your classes. What questions did the students have? What went well? What didn't go so smoothly? Revise the guided discovery lesson while it is fresh in your mind so that it will be correct, complete, and ready to use the following year.

<u>10. Bank and Share Your Guided Discovery Lessons</u>

You should not expect to use and/or create a guided discovery activity for every single math class. Discuss with your colleagues the possibility of several common-course teachers developing a few per year, either individually or as a group. At the end of that year your department would have a bank of field-tested guided-discovery activities which can be expanded in subsequent years.

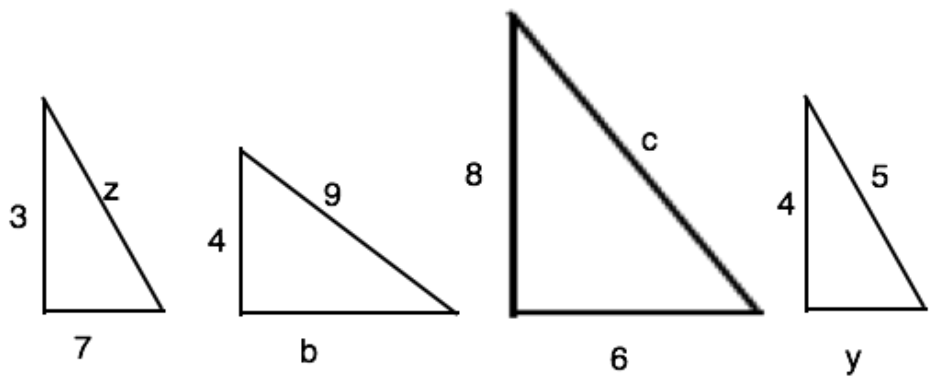
GUIDED DISCOVERY ACTIVITY 1: FIBONACCI MEETS PYTHAGORAS

Many of you have already studied the Fibonacci Sequence. This sequence begins with the terms 1 and 1. Each successive term is found by adding the two previous terms:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, . . .

In 1948, in a journal called Scripta Mathematica, author Charles Raine combined the Pythagorean Theorem and the Fibonacci Sequence to produce a fascinating result. The following steps will lead you to discover this relationship.

1. Find the length of the missing side in each of the following right triangles. If the answer is irrational, leave it in radical form.



2. Notice that the last two triangles have all integer sides. If a right triangle has all integer sides, the set of side lengths is called a Pythagorean triple. Do you know any other Pythagorean triples?

3. Notice that 3, 4, 5 and 6, 8, 10 are sides of a right triangle. The set 6, 8, 10 is a multiple of the set 3, 4, 5. Write three more Pythagorean triples such that one set is not a multiple of any of the other sets.

4. Now, look at the Fibonacci sequence presented in the opening paragraph. Write the next 6 terms of this sequence.

5. Examine the first 4 consecutive Fibonacci numbers: **1, 1, 2, 3**. Find the product of the first and the fourth number. Call it x.

6. Find twice the product of the middle two numbers. Call it y.

7. Find the sum of the squares of the middle two numbers. Call it z.

8. Show that $x^2 + y^2 = z^2$.

9. Examine the next four consecutive Fibonacci numbers: **1, 2, 3, 5**. Find the product of the first and fourth numbers. Call it r.

10. Find twice the product of the middle two numbers. Call it s.

11. Find the sum of the squares of the middle two numbers. Call it t.

12. Show that $r^2 + s^2 = t^2$.

13. Based on the work you have just completed, pick any four consecutive Fibonacci numbers. Use the procedure outlined above to form a Pythagorean triple. Verify that your numbers satisfy the Pythagorean Theorem.

14. Does this procedure always work? Let the first of any four consecutive Fibonacci numbers be represented by a, and the second by b. How can you represent the third and the fourth in terms of a and b?

15. Find the product of the first and fourth numbers in terms of a and b. Call this product x.

16. Find twice the product of the two middle numbers in terms of a and b. Call this y.

17. Find the sum of the squares of the two middle numbers in terms of a and b. Call this z.

18. Show that $x^2 + y^2 = z^2$.

.

19. Use the process you just discovered on the different sets of four consecutive Fibonacci numbers to complete the table.

4 CONSECUTIVE FIBONACCI NUMBERS	**PRODUCT OF THE 4 CONSECUTIVE FIBONACCI NUMBERS**	**ASSOCIATED PYTHAGOREAN TRIPLE**	**AREA OF THE RIGHT TRIANGLE FORMED BY THE TRIPLE**
1, 1, 2, 3			
1, 2, 3, 5			
2, 3, 5, 8			
3, 5, 8, 13			
5, 8, 13, 21			
$a, b, a+b, a+2b$			

20. ***Make a conjecture about the four consecutive Fibonacci numbers and the area of the right triangle formed when those numbers are used to create the sides.***

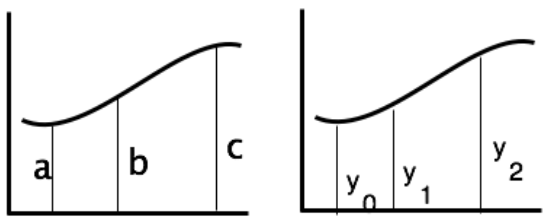
GUIDED DISCOVERY ACTIVITY 2:
FINDING THE AREA UNDER A CURVE USING THE TRAPEZOIDAL RULE

Prerequisite Knowledge

In this activity we will use equations of functions to approximate the area of regions bordered, in part, by curved boundaries. This is a major part of the study of calculus. It has many real-life applications, beyond finding area. To complete this activity successfully, you should be familiar with the use of subscripts in mathematics, the area of a trapezoid, and using your calculator to evaluate functions for different values of x.

The Use of Subscripts in Mathematics

A subscript is used to give a name to a variable. You could name the following three function heights a, b, and c or you could name them y_0, y_1, and y_2.



Keep in mind that the subscript is part of the name; it is used to keep track.

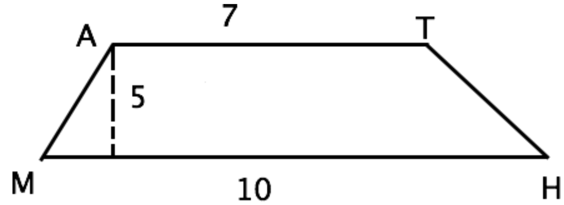
The Area of a Trapezoid

A trapezoid has two parallel sides called its **bases**, and a pair of nonparallel sides. The distance between the parallel sides is the **height** of the trapezoid. The area of a trapezoid with bases b_1 and b_2 and height h is

$$A = \frac{1}{2}h(b_1 + b_2)$$

No matter how the trapezoid is moved around in the plane, the bases are still the parallel sides and the height is still the perpendicular distance between the parallel sides.

Review the area formula by finding the area of the trapezoid MATH.



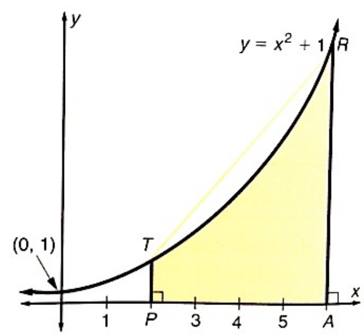
Evaluating f(x) for Different Values of x

You will need to evaluate functions f(x) for different values of x, and you should know how to do this on your calculator. Many calculators have efficient ways to do these substitutions, and you can even use tables of x and y values if your calculator has this feature. Review this with your group members by finding the value of Y_1 when x = 3 if

$$Y_1 = x^3 - 6x^2 + \frac{12}{x}.$$

Exploring An Approximation for the Area Under a Curve

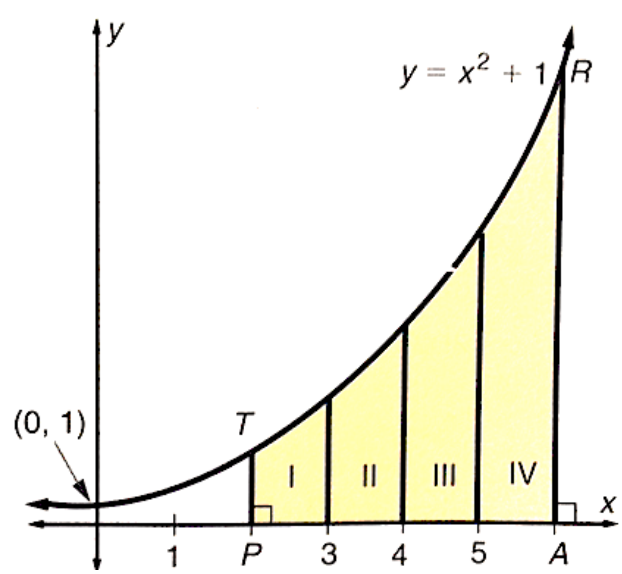
1. The shaded region below lies between two vertical lines, above the x-axis, and below the graph of $y = x^2 + 1$. Explain why the outline of the shaded region is not a polygon.



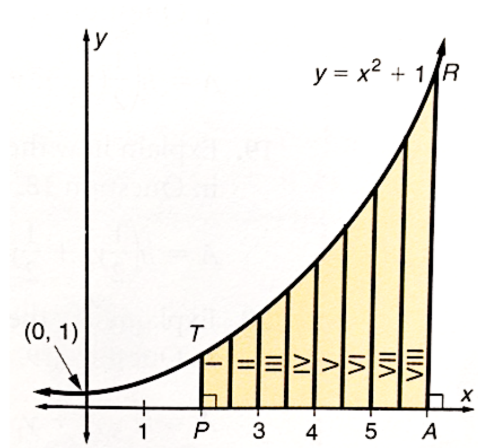
2. The shaded area is called the ***area under the curve***. It can be approximated using trapezoid TRAP. What is the measure of AP, the height of the trapezoid?
3. Which two sides of the trapezoid are parallel?
4. What is the length of base TP of the trapezoid? Explain how you found it.
5. What is the length of the longer base AR of the trapezoid?
6. What is the area of trapezoid TRAP?
7. Is the area of the trapezoid TRAP an over approximation or an under approximation of the area of the shaded region? Explain.

Improving the Approximation

8. The estimation of the area can be improved by dividing the area under the curve into several trapezoids, as shown. Find the area of the four trapezoids I, II, III, and IV.
9. Find the sum of the areas of the four trapezoids.
10. Does the sum you just found represent an over approximation or an under approximation of the shaded area?



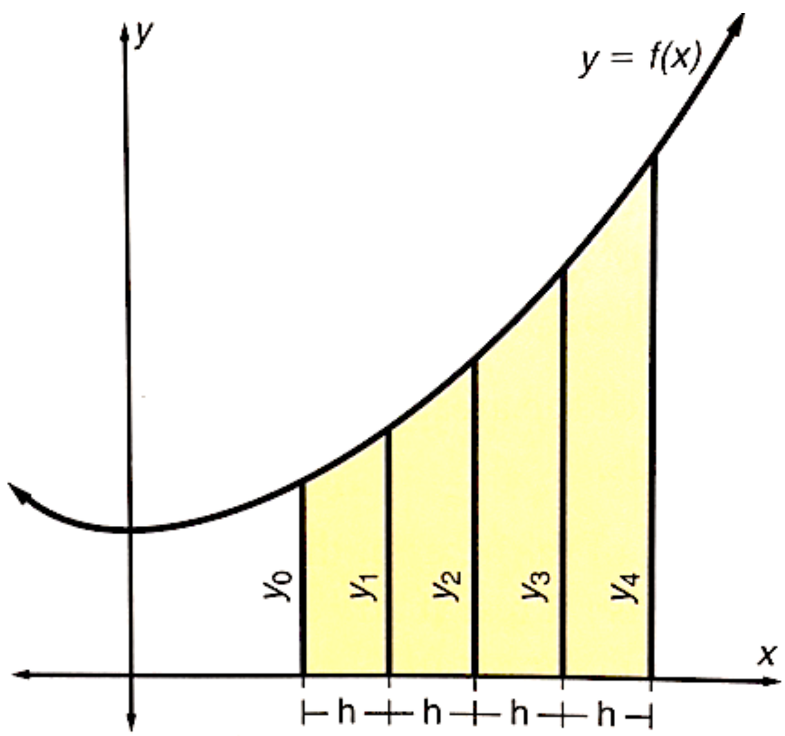
11. If the sum of the eight trapezoids, as shown, was computed, would it be a better or worse approximation of the actual shaded area than the approximation that used four trapezoids?



12. Find the height and the bases of the trapezoids I – VIII
13. Find the area of trapezoids I – VIII
14. Find the sum of the areas of trapezoids I - VIII.
15. Would using 200 trapezoids improve the approximation?
16. What is an advantage of using many trapezoids?
17. What is a disadvantage of using many trapezoids?

Using Algebra to Derive a New Formula

The following algebraic steps create a formula that simplifies the use of the trapezoids.



18. This step models the work you did in this activity. How many trapezoids are evident from this equation?

$$A = \frac{1}{2}h(y_0 + y_1) + \frac{1}{2}h(y_1 + y_2) + \frac{1}{2}h(y_2 + y_3) + \frac{1}{2}h(y_3 + y_4)$$

19. Explain how this equation was derived from the equation in #18.

$$A = h\left(\frac{1}{2}(y_0 + y_1) + (\frac{1}{2}(y_1 + y_2) + (\frac{1}{2}(y_2 + y_3) + (\frac{1}{2}(y_3 + y_4)\right)$$

20. Explain how this equation was derived from the equation in #19.

$$A = h\left(\frac{1}{2}y_0 + \frac{1}{2}y_1 + \frac{1}{2}y_1 + \frac{1}{2}y_2 + \frac{1}{2}y_2 + \frac{1}{2}y_3 + \frac{1}{2}y_3 + \frac{1}{2}y_4\right)$$

21. Explain how this equation was derived from the equation in #20.

$$A = h\left(\frac{1}{2}y_0 + y_1 + y_2 + y_3 + \frac{1}{2}y_4\right)$$

22. The formula derived in #21 is called the Trapezoidal Rule for Finding the Area Under a Curve. Notice how you can approximate the area by finding all of the y-values, taking half of the first and last y-value, finding a sum and then multiplying by h. This is more efficient than finding the area of each trapezoid separately. Write out the trapezoidal rule for eight trapezoids.
23. Find the area under the curve $y = x^2 + 1$ between $x = 2$ and $x = 6$ using eight trapezoids. Compare your answer to the answer you found in #14.

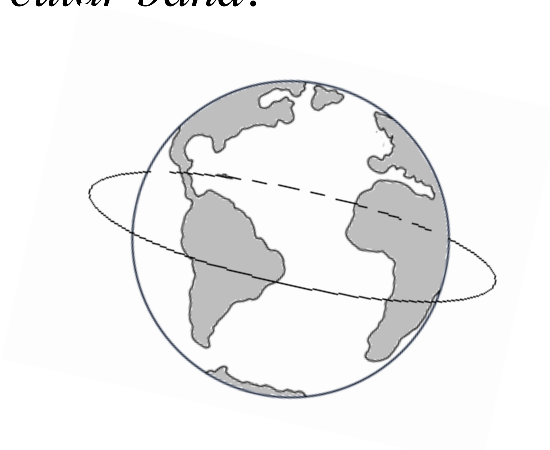
Practice What You Have Learned

24. If you used eight trapezoids to find the area under a curve from $x = 3$ to $x = 19$, what would the height of each trapezoid be?
25. Use the trapezoidal rule to approximate the area above the x-axis but under the curve $y = 2x^2 - 4$ between $x = 2$ and $x = 5$. Use three trapezoids.
26. Use the trapezoidal rule to approximate the area above the x-axis but under the curve $y = 2x^2 - 4$ between $x = 2$ and $x = 5$. Use six trapezoids.

GUIDED DISCOVERY ACTIVITY 3:

THE EFFECT OF THE LINEARITY OF THE RELATIONSHIP BETWEEN THE CIRCUMFERENCE AND RADIUS OF A CIRCLE

A steel band is placed around the earth, snugly fit at the equator. (The equator is approximately 25,000 miles in circumference.) The band is cut, and a 36-inch piece of string is spliced into the steel band. The new circular band is placed around the earth, centered off the earth's surface, so its center coincides with the center of the earth. A gap is created between the equator and this circular band.



1. Make a prediction about what could fit in this gap. A hair? An index card? How wide is this gap?

The teacher demonstrates the activity around a classroom globe. Each group, along with the teacher, finds the circumference of their circular object using the string, then ties that string-circumference to the 38-inch piece with a knot that will take up about two-inches of the string. The extended piece of string is placed on a table to form a concentric circle around the circular object. Students should attempt to make the string as circular as possible. Students measure the gap, g, that is, the difference between the measures of the radius of the string circle and the radius of the circular object. Data from the class is pooled and the following table is completed:

Table 1: Group Data

Circular Object	Gap, g
Nickel	
Cup	
Hubcap/cymbal	
Garbage can	
Cafeteria table	

2. Do you want to revise your prediction from question #1 based on the results your classmates found?

At this juncture, below you will find three different guided discovery activities:

- The arithmetic approach
- The algebraic approach
- The graphical approach

The three separate guide discovery lessons can be given in any combination you wish. For example, you may choose to do one in class, one for homework, one for extra credit. Or, they can do two in class and one for extra credit. You could also split the class into three groups and assign each group a different version of the guided discovery activity. Students could then report out their findings to the class as a group.

1. Make a prediction about what could fit in this gap. A hair? An index card?

2. How wide is this gap?

The Arithmetic Approach

3. Assume that the circumference of the earth is 25,000 miles. If one mile equals 5,280 feet, find the circumference of the earth in feet. ________________________

4. Find the circumference of the earth in inches. _______________________________

5. Using the formula $C=2\pi r$, find the radius of the earth in inches.

____

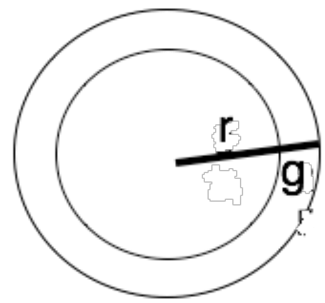
6. Add 36 inches to the circumference you found in question #4. This is the circumference of band after it has been enlarged.

____

7. Using the formula C=2πr, calculate the radius of the enlarged band.

____

8. The gap, labeled ***g***, is the difference between the radius of the earth, ***r,*** and the radius of the enlarged band. See the picture. Find this difference. ____________________



9. How does this difference compare to the gap found for each of the objects that your class measured and listed in Table 1 above?

____

10. A **conjecture** is a hypothesis—an educated guess. Make a conjecture about the size of the gap around *any* circle, based on all of these results.

11. Did the answer surprise you or did you expect it? ______________________________

<u>The Algebraic Approach</u>

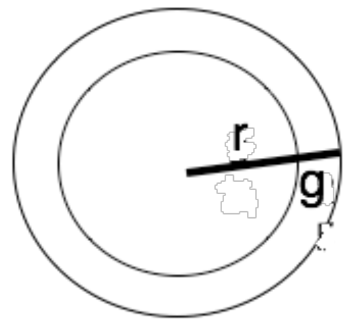
12. Let r represent the radius of the earth. Express the circumference of the earth, C, in terms of r.

____

13. Recall that the circumference was increased by 36 inches to form the enlarged band. Write an algebraic expression for "36 more than the circumference C".

____

14. Examine the picture of the concentric circle formed by the enlarged band around the earth. The radius of the earth is labeled *r* and the gap between the circles is labeled *g*.



Write an algebraic expression in terms of r and g for the radius of the large circle.

____

15. Write a circumference equation for the large circle, in terms of r and g, using the results from questions 13 and 14. *Reflection Point: Check this answer with your teacher before moving on to #16.* __________________________

16. Distribute 2π over r and g. Write the new equation.

____

17. In algebra, it is common to subtract equal quantities from both sides of an equation to solve it. Since C = 2πr, subtract C from the left side of the equation from question 16, and subtract 2πr from the right side. Write the new equation.

____

18. Solve the equation from question 17 for g. Round your answer to the nearest integer.

____

19. How does your answer to question 18 compare to your conjecture from question 10? Do they support each other or do they contradict each other?

The Graphical Approach

20. Write the equation for the circumference C of a circle in terms of its radius r.

____

21. Solve the equation from question 20 for r in terms of C. *Reflection Point: Check this answer with your teacher before moving on to question 22.*

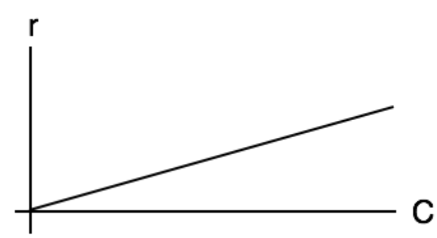
____

22. Examine the ***r*** and ***C*** axes below. Notice that the horizontal axis represents C (the dependent variable), which we commonly think of as ***x***, and the vertical axis represents r (the independent variable), which we commonly think of as ***y***.

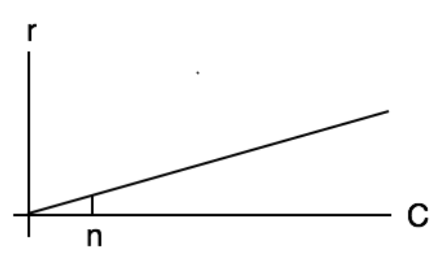
If you were graphing a line based on the equation from question 21 on these axes, what would be the slope of the line? _____________

23. What is the r-intercept of the line found in question 21? ________________

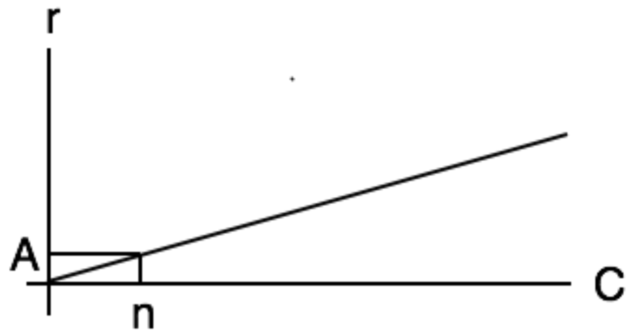
24. Examine this sketch of the line you found in question 21.



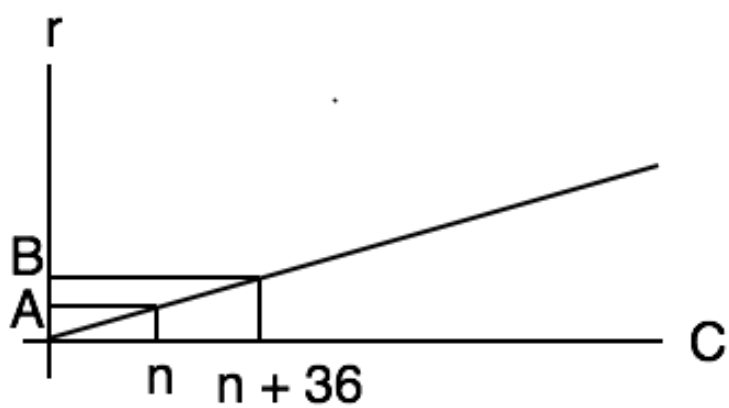
25. Let ***n*** represent the circumference of a nickel. Notice that n is plotted on the C axis below.



26. We will call the radius of the nickel A. Which axis is A on? _______.

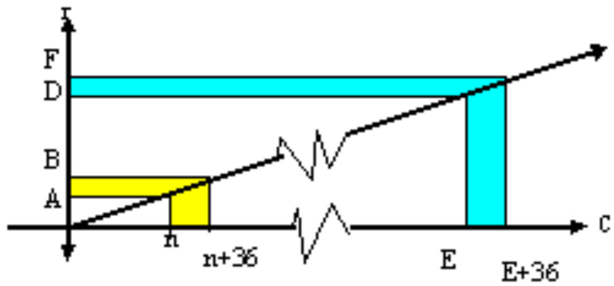


27. The circumference of the enlarged concentric circle you formed around the nickel is n + 36. Look where this is plotted on the C axis below. The radius of this circle is B as and B is plotted on the r axis. Since A and B are on the r-axis, what does B-A represent? _________________



28. Based on the data from Table 1, and your experience with the arithmetic and algebraic solutions, what is the value of B-A to the nearest integer? ______________________

29. Let E represent the circumference of the earth and let D represent the radius of the earth. Look at where they are plotted on the graph below. Also, let E+36 represent the circumference of the enlarged band and F represent the radius of the enlarged band. Study this on the graph below.



Can you explain why the graph is “broken” by the jagged lines in the middle?

30. Which of the following inequalities correctly represents the relationship between (B – A) and (F – D)?

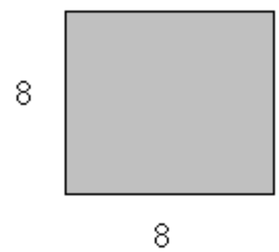
a) (B – A) > (F – D)
b) (B – A) = (F – D)
c) (B – A) < (F – D) _______________

31. What does your answer to question 30 tell you about the gap formed around any circle when 36 inches is added to the circumference?

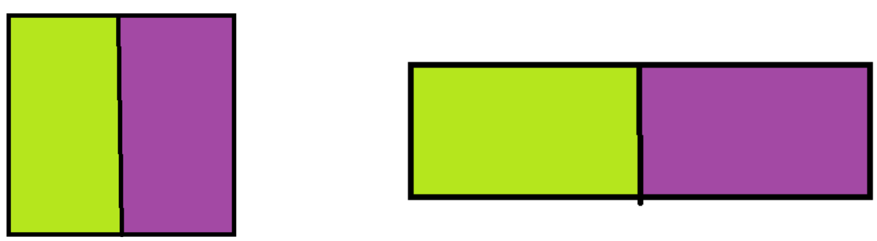
GUIDED DISCOVERY ACTIVITY 4:
AN EXPLORATION OF THE DISSECTION AND REARRANGEMENT OF SQUARES USING FIBONACCI NUMBERS

When finding the area of a square, multiply the length of its base by the length of its height. If the square is cut into different polygons and rearranged, the area should still be the same.

1. Find the area of the following 8 by 8 square.



2. The 8 by 8 square is **dissected** (cut up) into two congruent rectangles, and rearranged as shown below.



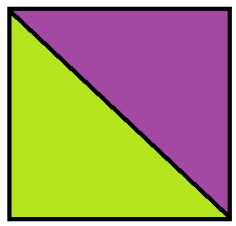
What is the length of the rectangle?______________________

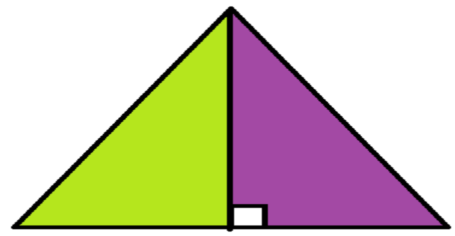
3. What is the width of the rectangle?____________________

4. What is the area of the rectangle?_____________________

5. What is the difference between the areas of the original 8x8 square and the rectangle?___________. Is this the answer you expected? Explain:

6. The same 8x8 square is cut along its diagonals into two triangles and then rearranged into one larger triangle, as shown below.





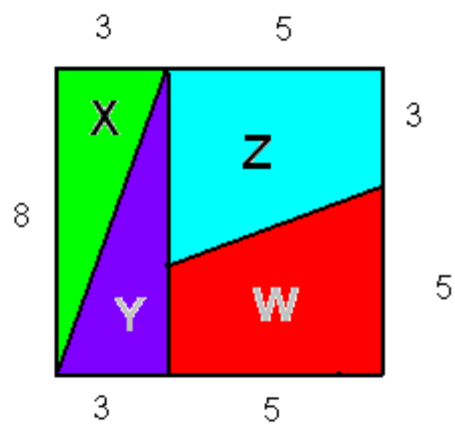
What is the length of the base of the triangle?_________________

7. What is the height of the triangle?________________

8. What is the area of the triangle?___________________

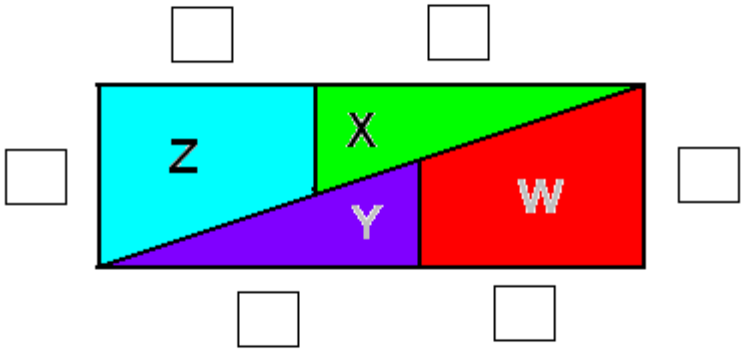
9. What is the difference between the areas of the original 8x8 square and the rectangle?___________. Is this the answer you expected? Explain:

10. The following 8x8 square is dissected into two congruent right triangles, X and Y, and two congruent trapezoids, Z and W.



What is the area of the square?____________________

11. The four polygons are rearranged to form the rectangle below. Fill in the boxes with the lengths of each segment, based on the lengths from the square.



12. What is the length of the rectangle?____________

13. What is the width of the rectangle?________________

14. What is the area of the rectangle?_____________________

15. What is the difference between the areas of the original 8x8 square and the rectangle?___________. Is this the answer you expected? Explain:

How is it possible that the areas are not equal? This guided discovery activity will help you learn why the area was able to "change" when the shapes were rearranged. This puzzle depends on the Fibonacci numbers:

1, 1, 2, 3, 5, 8, 13, 21….

16. The pictures from questions 10 and 11 are drawn "free-hand"; they are not drawn to scale. Perhaps this explains the discrepancy. Let's see what happens if we draw them to scale.

- Draw the square on ½ inch graph paper accurately.
- Carefully cut out the triangles and the trapezoids.
- Rearrange them to form the rectangle, and tape them together carefully.
- Does this help you explain what happened to the area? Explain.

17. List the first 10 Fibonacci numbers, f_1 through f_{10}.

____

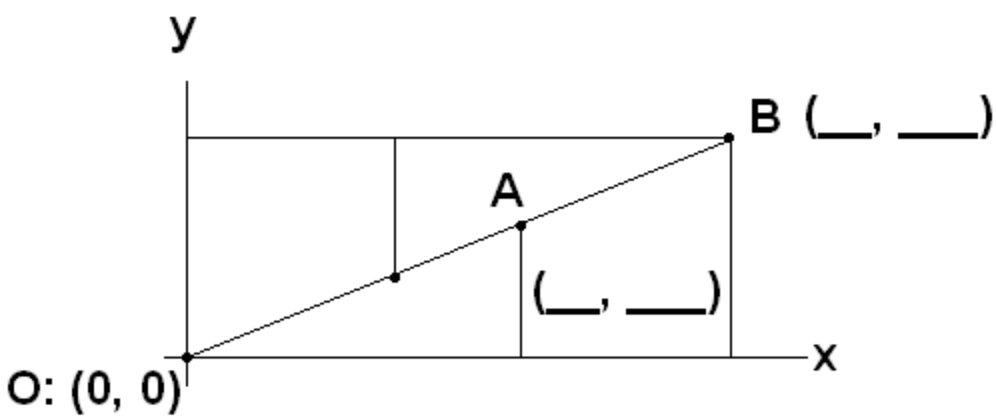
18. We are going to make fractions out of Fibonacci numbers. Fill in the following table. Write each of these ratios in fraction form, and then in decimal form:

Fraction in Fibonacci Form	Fraction in Numerical Form	Equivalent Decimal—Keep six decimal places

$\frac{f_1}{f_3}$		
$\frac{f_2}{f_4}$		
$\frac{f_3}{f_5}$		
$\frac{f_4}{f_6}$		
$\frac{f_5}{f_7}$		
$\frac{f_6}{f_8}$		

19. Look at the decimal representations for each fraction. Make a conjecture about the Fibonacci ratio $\frac{f_i}{f_{i+2}}$, as the subscripts get larger.

20. Look at the rectangle superimposed on the xy axes. Use the lengths of the segments form the rectangle in question 11 to find the coordinates of A and B.



21. Notice from question 11 that all of the given side lengths are Fibonacci numbers. Find the slope of OA as a fraction. ___________________

22. Find the slope of OA as a decimal—keep six decimal places.__________________

23. Find the slope of AB as a fraction.______________________________

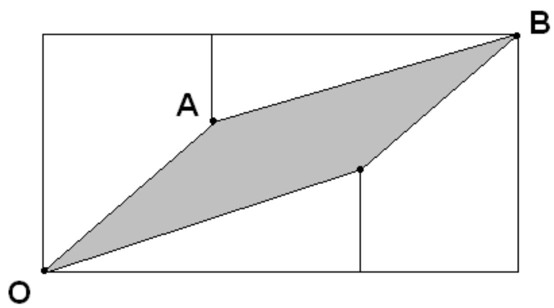
24. Find the slope of AB as a decimal—keep six decimal places.__________________.

25. Compare your slopes to your answers to questions 22 and 24 above. Does the slope of OA equal the slope of AB?_______________

26. Is OAB a straight line?__________________

27. Which line segment, OA or AB, is steeper?________________________

28. How does this picture use slope to explain the area discrepancy?



EXTENSION 1: What shape is quadrilateral OABC?
EXTENSION 2: Look at the original square from the beginning of the problem. Draw a square composed of the two triangles and trapezoids, but label the sides with the Fibonacci numbers 5, 8 and 13 taking the respective place of 3, 5, and 8. Create the new rectangle and compare the areas of the square and the rectangle. How can you explain this area discrepancy?

GUIDED DISCOVERY EXAMPLE 5
THE AREA OF A TRIANGLE

1. The formula for the area of a rectangle with base b and height h is ________________.
2. What is the area of the rectangle in Figure 1? _________________

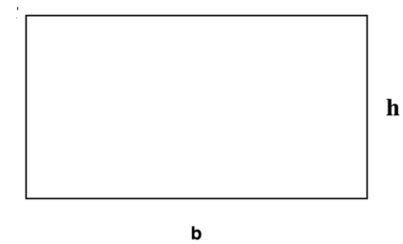


Figure 1

3. Rectangle ABCD has right triangle ABE shaded in as shown in Figure 2.

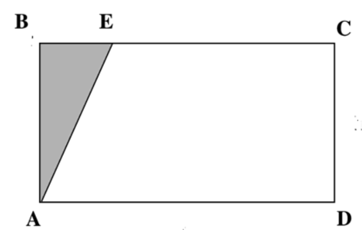


Figure 2

4. Shaded right triangle ABE is cut off and translated horizontally to the right so that sides AB and CD coincide, as shown in Figure 3. The points are relabeled. The opposite sides are parallel. What is the name of the quadrilateral WXYZ? ________________________

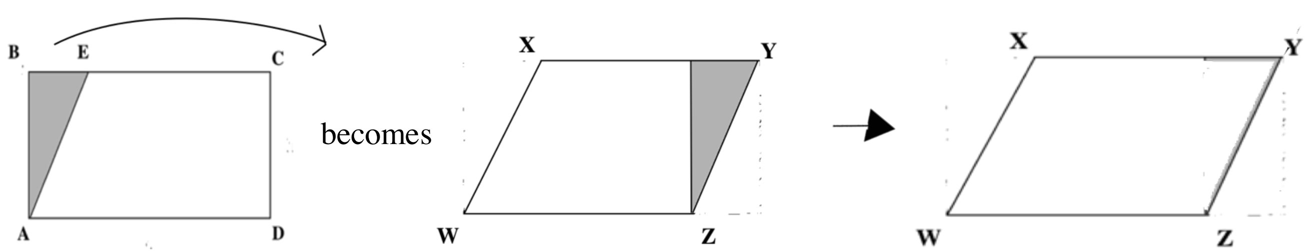


Figure 3

5. Does the area of the newly formed quadrilateral WXYZ have the same area as the original rectangle? Explain. ____

6. Diagonal XZ is drawn as shown in Figure 4, cutting the quadrilateral in half. The height, shown by the dashed line, of shaded triangle WXZ, is _______ and the base is _________. What is the area of triangle WXZ? ______________

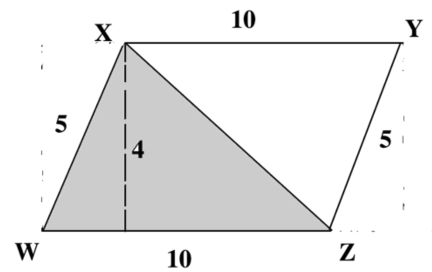


Figure 4

7. Did you use the side length of 5 used to compute the area? ____________
8. Diagonal HK is drawn as shown in Figure 5, cutting parallelogram GHJK in half. The height of this triangle is _______ and the base is _________. What is the area of triangle GHK? ______________

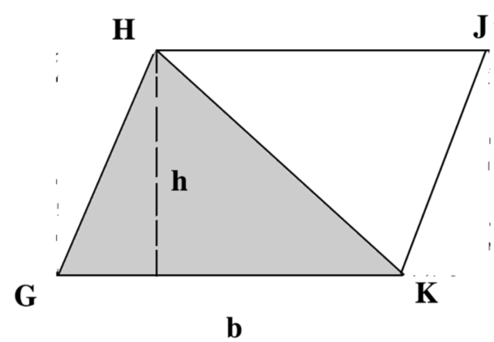


Figure 5

9. Now look at a triangle on its own, when it is not half of a parallelogram. What is the area of the triangle in Figure 6? ____________________

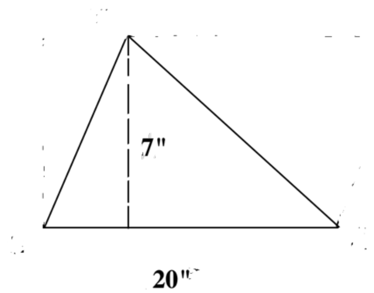
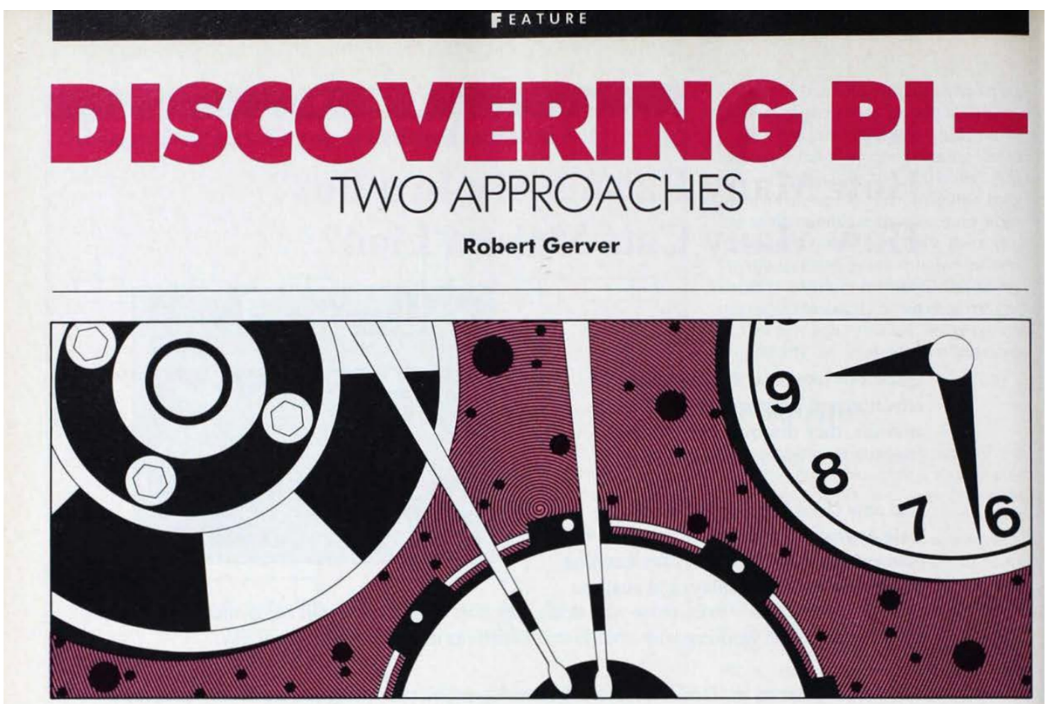


Figure 6

10. What is the formula for the area of a triangle with height h and base b?

____

THE APRIL 1990 ISSUE OF THE NCTM'S *ARITHMETIC TEACHER* HAS A GUIDED DISCOVERY LESSON ON DISCOVERING PI. NCTM Members can download it at nctm.org. Look under Legacy Journals.

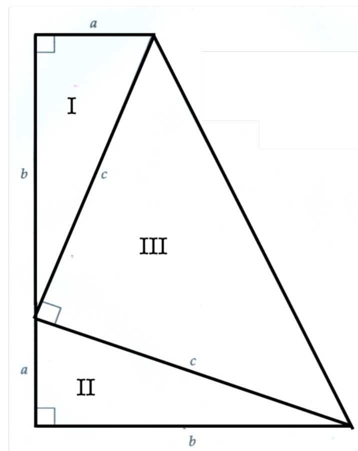


GUIDED DISCOVERY ACTIVITY 6:
GUIDED DISCOVERY: PROVING THE PYTHAGOREAN THEOREM

A right triangle has legs a and b and hypotenuse c. We are going to look at a relationship between these three sides that holds for all right triangles.

1. A right triangle has legs 4 and 17. Find the area of the triangle.____________

2. A triangle has legs a and b. Express the area of the triangle in terms of a and b. ____________



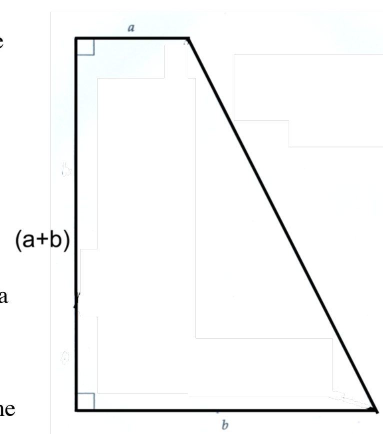
3. In the picture on the right, find the area of right triangle I. _____________

4. Find the area of right triangle II.______________

5. Find the area of right triangle III.___________

6. Eliminate the sides with length c in the picture. Since the quadrilateral has only one pair of parallel sides, the quadrilateral is a _________________.

7. To find the area of the quadrilateral, you can add the area of the three right triangles. Combine like terms and express this area in simplest form.
____



8. Express the measure of the height of the trapezoid on the right in terms of a and b._____________

9. What is the length of the larger base of the trapezoid?_______

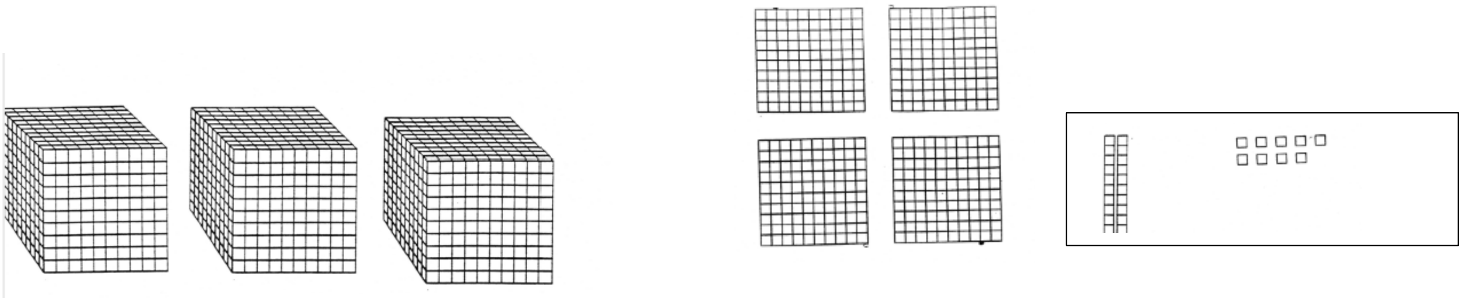
10. What is the length of the smaller base of the trapezoid?______________

11. What is the area of the trapezoid expressed in terms of a and b?________________________________

12. Square the binomial (a + b) your expression from question 11, and rewrite your answer to question 11 with the trinomial you found after multiplying (a + b) by (a + b).
____

GUIDED DISCOVERY ACTIVITY 7:
DIVISIBILITY - The Rule of 9

1. Examine the Base Ten Blocks model below. What number is modeled here? ______



CUBES **FLATS** **LONGS** **UNITS**

2. The ***cube*** represents 1000. How is the expression (1000 – 1 + 1) equivalent to 1000?

Above, you were introduced to the concept of a zero pair. Notice that (-1 + 1) has a value of zero and can be combined with any number or expression without changing its value.

3. What is the multiple of 9 that is closest to 1000?________
   Show that your number is indeed a multiple of 9.

4. You should have found that 999 is the multiple of 9 closest to 1000. Use the Associative Property (the grouping property) in 1000 -1 + 1 to introduce the multiple that you found above (999) into the expression?

5. It is clear that 1000 = **999** + 1. In the Base Ten Blocks figure above, cross off one of the units in each of the thousands *cubes.* Each of these blocks will now represent 999. We still want the model to represent 3429, so draw three units blocks under the figure to show the three units you removed from the thousands cube.

6. The ***flat*** represents 100. How is the expression (100 -1 + 1) equivalent to 100?

Above, you again used a zero pair. Notice that (-1 + 1) has a value of 0 and can be combined with 100 without changing its value.

7. What is the multiple of 9 that is closest to 100? __________ Show that your number is indeed a multiple of 9.

8. Use the Associative Property (the grouping property) in 100 – 1 + 1 to introduce the multiple that you found above into the expression.

9. It is clear that 100 = 99 + 1. In the Base Ten Blocks figure above, cross off one of the units in each of the hundreds *flats.* Each of these blocks will now represent 99. We still want the model to represent 3429, so draw 4 units blocks under the figure to show the 4 units blocks you removed from the flats.

10. The ***long*** represents 10. How is the expression (10 -1 + 1) equivalent to 10?

Above, you again used a zero pair. Notice that (-1 + 1) has a value of 0 and can be combined with 10 without changing its value.

11. What is the multiple of 9 that is closest to 10? ______ Show that your number is indeed a multiple of 9.

12. Use the Associative Property (the grouping property) in 10 -1 + 1 to introduce the multiple that you found above into the expression.

13. It is clear that 10 = 9 + 1. In the Base Ten Blocks figure above, cross off one of the units in each of the tens *longs.* Each of these blocks will now represent 9. We still want the model to represent 3429, so draw 2 units blocks under the figure to show the 2 units blocks you removed from the longs.

14. Look at the altered base tens blocks figure. Notice what is now represented:

999 999 999 99 99 99 99 9 9

and

1 1 1 (from the thousands cubes)
1 1 1 1 (from the hundreds flats)
1 1 (from the tens longs)
1 1 1 1 1 1 1 1 1 (from the original units place)

Therefore, 3429 =999 + 999 + 999 + 99 + 99 + 99 + 99 + 9 + 9 + (3 + 4 + 2 + 9) which can also be written as

3429 =999 + 999 + 999 + 99 + 99 + 99 + 99 + 9 + 9 + (18)

Is 3 + 4 + 2 + 9 a multiple of 9?_____________
Is each addend a multiple of 9?_____________

15. How can you use this information to conclude that 3429 is a multiple of 9?

16. In your own words, write the rule to test for divisibility by 9.

Guided Discovery Activity 8: Division of Fractions

1. **Determine the value of x in the following equation.**
   Express the value as a fraction.
   $\frac{16}{27} \div \frac{4}{9} = x$

2. **Determine the value of y in the following equation.**
   Express the value as a fraction.
   $\frac{3}{4} \div \frac{7}{5} = y$

3. **Find a, b, c, and d:** $16 \div 4 = a$ $27 \div 9 = b$
   $3 \div 7 = c$ $4 \div 5 = d$

4. **Is x =** $\frac{a}{b}$**? How is that possible? You "divided across"? Most likely, that is NOT the method you learned for dividing fractions!!!! Does this always work??**

5. **Let's go back to the equation in #2 above:** $\frac{3}{4} \div \frac{7}{5} = y$
   Is y = $\frac{c}{d}$?
   Maybe we need to do a little rewriting.

6. **In # 3 above, you should have found that c =** $\frac{3}{7}$ **and d =** $\frac{4}{5}$**.**
   Write the numerical value of $\frac{c}{d}$ **as a complex fraction in the form** $\frac{\frac{?}{?}}{\frac{?}{?}}$ **.**

7. **What can you multiply the denominator of the fraction by to get 1 in that denominator?**

8. **Multiply the numerator and the denominator of the complex fraction by this number. Simplify where possible. What do you notice?**

9. **Let's return back to $\frac{3}{4} \div \frac{7}{5}$. Write the answer as a simplified fraction.**

10. **Return to the division statement. Rewrite each fraction in an equivalent form with common denominators.**

$$\frac{3}{4} \div \frac{7}{5} = \frac{e}{f} \div \frac{g}{h}$$

11. **Now, using the new equivalent fractions ($\frac{e}{f}$ and $\frac{g}{h}$), divide across from left to right as shown here:**

$$\frac{e}{f} \div \frac{g}{h} = \frac{e \div g}{f \div h}$$

Express the answer as a simplified fraction. What do you notice?

GUIDED DISCOVERY ACTIVITY 9: THE MIDDLE ORDINATE OF A CIRCLE

Context: Automobile Accident Skid Marks

Source: *Financial Algebra*, Gerver and Sgroi, NGL/Cengage Learning, 2021

Background Information

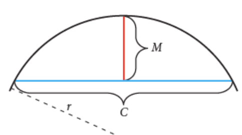
When a car enters a skid and the brakes lock (or lock intermittently), the driver cannot control the steering. Therefore, the skid is usually a straight line. The vehicle continues to move straight ahead as the brakes lock, making the tire marks straight. When the vehicle slips sideways while at the same time continuing in a forward motion, the tire marks appear curved. These are called yaw marks.

Taking skid and yaw measurements, as well as other information from the scene, can allow accident reconstructionists to compute the speed of the car when it entered the skid. The formulas used are often presented in court and are recognized for their strength in modeling real-world automobile accidents.

At the scene of an accident where a car leaves curved tire marks (yaws) the minimum speed can be determined from the data available by measuring the yaw marks. If S is the minimum speed, f is the drag factor, and r is the radius of the arc of the yaw mark, the most basic formula is

$$S = \sqrt{15fr}$$

To identify a radius, you must be able to find the center of the circle of which the arc is part. Here is how reconstructionists do that. First, they select two points on the outer rim of the arc and connect them with a chord. A chord is the line segment that connects two points on an arc or circle as shown.



The center of the chord is located and a perpendicular line segment is drawn from that center to the arc, creating a right angle. This short line segment is the middle ordinate.

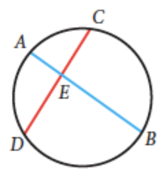
Reconstructionists use the following formula to determine the radius.

$$r = \frac{C^2}{8M} + \frac{M}{2}$$

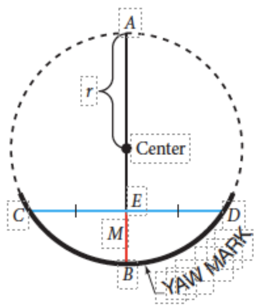
Where r is the radius of the yaw arc, C is the length of the chord, and M is the length of the middle ordinate.

But where did that formula come from? This guided discovery activity develops the formula using geometric relationships learned in a high school Geometry course.

Prerequisite Knowledge: Setting the stage for the guided discovery



The formula used to determine the radius of the yaw mark arc is derived from a geometric relationship about two intersecting chords in a circle. In the figure, chords $\overline{AB}$ and $\overline{CD}$ intersect at point E in the circle. The product of the two segment lengths making up chord $\overline{AB}$, AE X EB, is equal to the product of the two segment lengths making up chord $\overline{CD}$, CE X ED.



In the figure at the left, the yaw mark is continued as a dotted line to form a complete circle. A chord is drawn connecting two points on the yaw mark. The middle ordinate is also drawn. The length of the middle ordinate is M and the length of the chord is CD. The middle ordinate cuts the chord into two equal pieces with each half of the chord CD/2 units in length. The radius of the circle has length r as shown in the diagram. Applying the property of two intersecting chords in this diagram you get AE X EB = CE X ED.

With that reviewed, work through these guided steps to discover the formula for the radius of the yaw arc in terms of the chord length and the middle ordinate length.

a) **From the diagram, CE = CD/2, ED = CD/2, and EB = M. You need to determine the length AE. Write an expression for AB in terms of r. ________________________**

b) **Notice that AE = AB – EB. Write an algebraic expression that represents the length AE in terms of r and M. ______________________________**

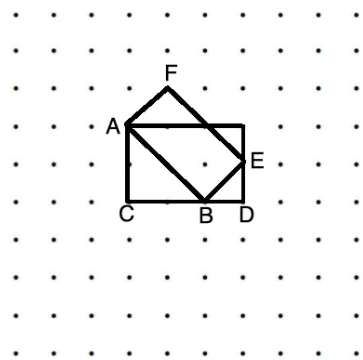
c) **Recall the property of two intersecting chords in a circle.**
Applied here, AE X EB = CE X ED. Write the algebraic expression for this equation in terms of r, M, and chord length CD. ________________________

d) **Use your results from part c) above. Simplify the side of the equation that represents the product of the segments of chord $\overline{CD}$. Write the new equation.**
____

e) **Solve the equation for r by isolating the variable r on one side of the equation. Show your work. Compare your answer with the radius formula on the previous page.**

GUIDED DISCOVERY ACTIVITY 10: MULTIPLICATION OF SQUARE ROOT RADICALS

In the figure below, the dots are 1 unit apart horizontally and vertically. For all computations, leave non-integer answers in radical form.



1) ΔABC is a right triangle with hypotenuse of length *L.* Use the Pythagorean Theorem to determine *L.*

____

2) ΔBDE is a right triangle with hypotenuse of length *W*. Use the Pythagorean Theorem to determine *W*.

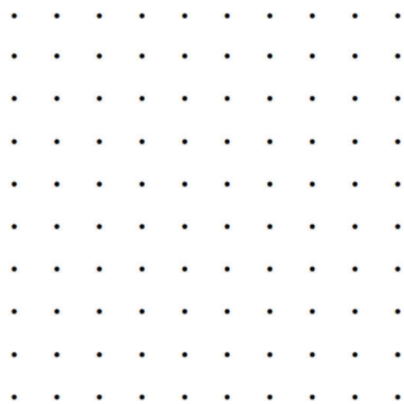
____

3) Find the area of rectangle AFEB by counting the number of square units covered by the rectangle.

____

4) How do steps #1 - #3 illustrate that $\sqrt{8} \times \sqrt{2} = \sqrt{16}$

5) Draw a figure on the grid below that is similar but not congruent to that given at the beginning of this activity. Label it in the same way.



6) ΔABC is a right triangle with hypotenuse of length *L.* Use the Pythagorean Theorem to determine *L.*

____

7) ΔBDE is a right triangle with hypotenuse of length *W*. Use the Pythagorean Theorem to determine *W*.

____

8) Find the area of rectangle AFEB by counting the number of square units covered by the rectangle.

____

9) Write a radical equation based upon the lengths of the segments and the area of the rectangle found in #6 - #9.

____

10) Based on your work here, write a “rule” for multiplying square root radicals. Explain your reasoning.

GUIDED DISCOVERY ACTIVITY 11: The Rule of 72

Albert Einstein said that compound interest was "the most powerful thing I have ever witnessed." Work through the following exercises to discover a pattern Einstein discovered.

To do this activity, you will need to use the compound interest formula as shown here.

Compound Interest Formula

$$B = P\left(1 + \frac{r}{n}\right)^{nt}$$

where B = ending balance
P = principal or original balance
r = interest rate expressed as a decimal
n = number of times interest is compounded annually
t = time in number of years

a.
Suppose that you invest $2,000 at a 1% annual interest rate. Use your calculator to input different values for t in the compound interest formula. What whole number value of t will yield an amount closest to twice the initial deposit?

b.
Suppose that you invest $4,000 at a 2% annual interest rate. Use your calculator to input different values for t in the compound interest formula. What whole number value of t will yield an amount closest to twice the initial deposit?

c.
Suppose that you invest $20,000 at a 6% annual interest rate. Use your calculator to input different values for t in the compound interest formula. What whole number value of t will yield an amount closest to twice the initial deposit?

d.
Albert Einstein noticed a very interesting pattern when an initial deposit doubles. In each of the three examples above, multiply the value of t that you found times the percentage amount. For example, in part a, multiply t by 1. What do you notice?

e.
Einstein called this the Rule of 72 because for any initial deposit and for any interest percentage, 72 ÷ (percentage) will give you the approximate number of years it will take for the initial deposit to double in value. Einstein also said, "If people really understood the Rule of 72 they would never put their money in banks."

Suppose that a 10-year-old has $500 to invest. She puts it in her savings account that has a 1.75% annual interest rate. How old will she be when the money doubles? 51 years old

GUIDED DISCOVERY ACTIVITY 12:

SYSTEMS OF LINEAR EQUATIONS: Auto Expense and Depreciation

Celine bought a new car for $33,600. She made a $4000 down payment and pays $560 each month for 5 years to pay off her loan. She knows from her research that the make and model of the car she purchased straight line depreciates to zero over 10 years. Write an expense and depreciation equation for this situation. Graphs this system of two equations on the same axes and interpret the results.

1. What was Celine's initial payment? _________________

This amount represents her first investment in the car when x=0 months. This will represent the y-intercept of the expense function.

2. How much does Celine pay per month? _______________

3. How could the ratio of the change in the total expense per month be expressed as a fraction? __________________

This will represent the slope of the expense function.

4. Use the information from above to express the linear expense function:
y = _____________________________

5. The time x is in months rather than years. Since Celine's car will totally depreciate after 10 years, how many months will it take for the value of this car to reach zero? _______________.

This amount will represent the x-intercept of the depreciation equation.

6. What was the purchase price of Celine's car? ________________

This amount will represent the y-intercept of the depreciation equation. Why?

7. What is the value of the ratio of the original car cost over the total number of months it will take for the car to depreciate? What does that value represent?

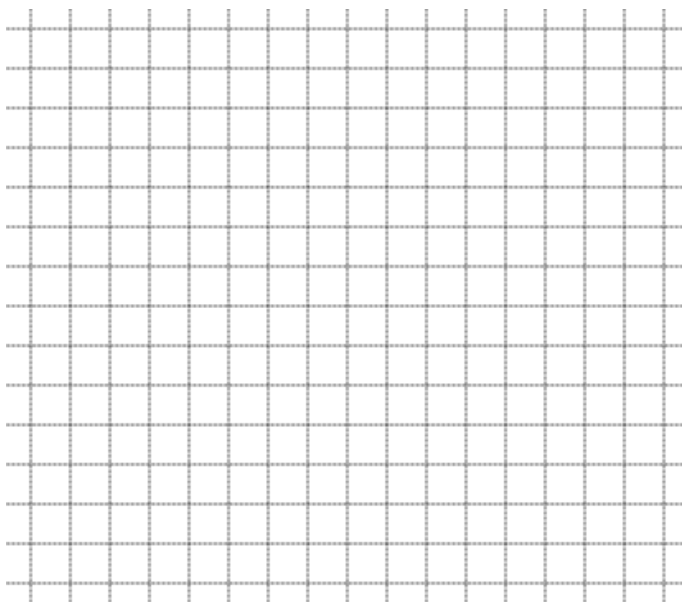
8. To calculate the slope of the depreciation equation, you will use the intercepts (0, 33600) and (120,0). In the context of this problem, what do these intercepts represent?

9. Use the coordinates of the intercepts to determine the slope of the depreciation equation? Where have you seen that number before?

10. In the context of this problem, what does the slope represent? Why is it negative?

11. Write the depreciation function. _________________

12. Graph of this system:



The lines intersect at the point
(35.24, 23733.33). In the context of this depreciation problem, what does this point of intersection represent?

What story does the graph tell on the domain $x < 35.24$?
What story does the graph tell on the domain $x > 35.24$?

GUIDED DISCOVERY ACTIVITY 13: EXPLORING POLYNOMIALS

Polynomials can be written in a variety of ways:

in expanded form: $p(x) = x^4 + 4x^3 - x^2 - 16x - 12$

in factored form: $p(x) = (x + 2)(x - 2)(x + 1)(x + 3)$

and in a third form. In this guided discovery, you will learn how to nest a polynomial. The information will be used to extend your knowledge of polynomials.

Examine the polynomial $p(x) = x^4 + 4x^3 - x^2 - 16x - 12$

1) Is there any factor common to each term of this polynomial other than 1. If so, name it. ____________

2) Find a factor common to all but one of the terms. _______

3) Use your answer from #2. Factor out the common factor (we will call it factor1 as shown below) and write the new form of the polynomial:

( )factor1 +constant1

____

4) Now, examine the expression within the parentheses. Find a common factor common to each term. If that isn't possible, find a factor common to all but one of the terms within the parentheses. ______________

5) Use your answer from #4. Factor out the common factor (we will call it factor2 as shown below) and write the new form of the polynomial:

(( )factor 2 + constant2)factor1 +constant1

____

6) Now examine the expression contained within the inner most set of parentheses. Find a factor common to all but one of the terms in those parentheses. ____________

7) Use your answer from #6. Factor out the common factor (we will call it factor 3 as shown below) and write the new form of the polynomial:

((( )factor3 + constant 3)factor 2 + constant2)factor1 +constant1

---

8) Since there are no other factors common to the two terms remaining in the inner most set of parentheses, the nested form of this polynomial is now complete. Recall your work with order of operations. Evaluate the nested form of the polynomial when x=5. Check your answer with what you would get had you evaluated the polynomial in standard form.

p(5) = ___________________________

9) Write out the exact steps that you used to evaluate this polynomial in nested form.

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10) Test your understanding: $r(x) = x^4 - 3x^3 - 15x^2 + 19x + 30$
Write the nested form of the polynomial and then use the nested form to evaluate $r(4)$. Evaluate the polynomial using the standard form and compare your results. How does nested form making evaluating a polynomial by hand easier?

ANSWERS:

1) No
2) x
3) $(x^3 + 4x^2 - x - 16)x - 12$
4) x
5) $\left((x^2 + 4x - 1)x - 16\right)x - 12$
6) x
7) $\left(\left((x + 4)x - 1\right)x - 16\right)x - 12$
8) (((5+4)5-1)5-16)5-12
9) Go from left to right in the standard form: $p(x) = x^4 + 4x^3 - x^2 - 16x - 12$
5 + 4 = 9; 9 x 5 = 45; 45 -1 = 44; 44x 5 = 220; 220 – 16 = 204; 204x 5 = 1020;
1020 – 12 = 1008.

10) (((x-3)x-15)x+19)x+30 = r(x)

$r(4) =$ = (((4-3)4-15)4+19)4+30= -70

$r(4) = 4^4 - 3 \cdot 4^3 - 15 \cdot 4^2 + 19 \cdot 4 + 30 = -70$

Guided discovery with magic squares

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Creating and Using Guided-Discovery Lessons